**Unit 5: Uniqueness of Solutions**

Goals/Rationale

The issue of whether or not a solution graph ever becomes an equilibrium solution was raised explicitly in homework problem 4 Unit 1 and it may have come up at other times with the logistic model. As such, students may have already been thinking informally about what experts see as an issue of uniqueness. This unit focuses on the uniqueness theorem and excludes treatment of existence. We focus only on uniqueness because it is slightly easier to motivate with a context, has some graphical appeal as the conditions connect to graphical reasoning (both slope fields and graphs of dh/dt vs. h). This issue of uniqueness (here understood informally to be concerned with whether or not two solutions touch or cross) is set in the context of a helicopter as it nears the ground. Two models are proposed and students explore what each model predicts about whether the helicopter will land. Each model has equilibrium solution h(t)=0 and hence landing is commensurate with two solutions touching.

**Page 5.1-5.3 – Proposed Paths of Descent**

Implementation Notes and Student Thinking

*Problem 1* – The goal of problem 1 is for students to develop some informal ways of reasoning about solutions to the two proposed models. For parts (a) and (b) we expect students to reason something like the following: For both equations the derivative is negative so the height will be decreasing. At this point there is no reason for them to suspect that one models predicts that the helicopter will land and the other doesn’t. There is no reason to come to consensus at this point, but only to solicit different ideas and nascent reasoning.

*Problem 2* – It is important that students are clear that h(t)=0 is a solution function for both equations, otherwise the issue of graphs of solution functions touching is lost. Thus parts (a) and (b) ask the same question in two different ways. The research literature has documented that some high school students do not consider constant functions to be legitimate examples of function because the dependent variable does not vary. Your students are likely more sophisticated, but it is good to be aware of this potential misconception.

*Problem 3* - Using the GeoGebra app, students will likely realize that there might be a difference in what the two differential equations predict. After zooming in on the slope field students will notice that the slopes for dh/dt = -h are much flatter near h=0 than they are for dh/dt = -h^(1/3). Encourage students to explain why this is the case.

*Problems 4 & 5* – Here students actually solve the IVPs and interpret the results in terms of the helicopter scenario. Students will inevitably have some difficulty with the closed form solution to dh/dt = -h^(1/3). In particular, once h is zero, then the solution will remain zero for all time. In terms of the context, this makes sense: Once the helicopter lands it stays on the ground.

Moreover, if you know that the helicopter is on the ground and you know that it was previously in the air, you are unable to know or accurately predict it’s prior descent. Imagine you've got a video tape of helicopter sitting on a platform and you realize you are in the middle of the video tape. And you can rewind and the helicopter maybe takes off. For any constant c you can define a solution that lands and stays on the ground for any time so that in backwards time you don't know when it's going to take off. When you are rewinding the tape, it could take off at any time so you don't know which solution you're on when you have that particular starting point in the tape. Whereas if you start in the air you know exactly which solution you're on.

*Problem 6* – This problem makes use of a graph of an autonomous differential equation first introduced in Unit 1 homework. Students should be able to argue that the partial derivative of dh/dt does not exist when h=0 for dh/dt = -h^(1/3) but it does for dh/dt = -h. This leads to the formal statement of the uniqueness theorem in terms of the Lipshitz condition.

**Page 5.4 – the Uniqueness Theorem**

Implementation Notes and Student Thinking

Students will need some help unpacking the theorem and making the connection to their graphs of dh/dt vs. h. Problem 7 underscores what you can conclude when the conditions are met and Problem 8 underscores what can and cannot conclude when the conditions of the theorem are not met.

**Notes for Personal Reflections on Unit 5**